PHY456 Lecture 4:
Classical Rotations, Group Theory, Quantum Rotations Review

<u>Spacetime Symmetry</u>: operations acting on dynamical variables $(\vec{r}_i, t)$ such that the equations of motion are left invariant.

- Time (E)
- Space $(\vec{p})$
- Rotation $(\vec{M})$
- Galilean boosts

This lecture: rotations in $\mathbb{R}^3$ and $\mathbb{C}^2$; invariance under $\vec{r}_i \to \vec{r}_i{}'$ $(\forall i)$ gives rise to angular momentum conservation.

$\boxed{CM}$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} & \cdots & \end{pmatrix}}_{\substack{3\times 3 \\ \text{orthogonal} \\ \text{matrix}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\to$ preserves vector length $|\vec{r}\,'| = |\vec{r}|$.

SO(3) : group of matrices $\det(M) = 1$ and 3×3 orthogonal.

"Special ($\det = 1$) Orthogonal ($O^T O = \mathbb{1}$) 3×3 (matrix)"

In CM: rotation around $\hat{z}$ axis by angle α

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For small α, can approximate

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

thus $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \sim \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \alpha y \\ -\alpha x \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \alpha \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left(1 + \alpha \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mathcal{O}(\alpha^2)$$

Any rotation on finite α = Π (rotations on infinitesimal ones)

Recall $\quad e^A = \lim_{N\to\infty} \left(1 + \frac{A}{N}\right)^N$, hence

$$\lim_{N\to\infty}\left(1+\frac{\alpha}{N}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\right)^N = e^{\alpha\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)}$$

$y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$ "generator of rotations"

$\Longrightarrow$ rotation on finite α along $\hat{z}$ $= e^{\alpha\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)}$

$$= 1+\alpha(\ )+\frac{\alpha^2}{2}(\ )^2+\ldots+\frac{\alpha^n}{n!}(\ )^n$$

We say rotations (around $\hat{z}$) are generated by the action of $y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$; meaning, apply a small rotation infinitely many times gets a finite rotation.

$\boxed{\text{QM}}$

$\frac{\partial}{\partial x}\to\frac{i}{\hbar}\hat{P}_x$ $\qquad x\to\hat{x}$

$\frac{\partial}{\partial y}\to\frac{i}{\hbar}\hat{P}_y$ $\qquad y\to\hat{y}$

the operator $\hat{x}$, or position, NOT the unit vector.

Hence $y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\to\frac{i}{\hbar}(\hat{y}\hat{P}_x-\hat{x}\hat{P}_y)$

$$\to\frac{i}{\hbar}(\hat{x}\hat{P}_y-\hat{y}\hat{P}_x) = \frac{i}{\hbar}\hat{L}_z$$ } angular momentum operator.

Similarly: $\hat{L}_x = \hat{y}\hat{P}_z-\hat{z}\hat{P}_y$

$\hat{L}_y = \hat{z}\hat{P}_x-\hat{x}\hat{P}_z$

$\hat{L}_z = \hat{x}\hat{P}_y-\hat{y}\hat{P}_x$

We may define

$$\boxed{\begin{aligned} \hat{L}_k &\equiv \epsilon_{k\ell m}\,\hat{x}_\ell\,\hat{p}_m \\ \hat{L}^2 &= \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \end{aligned}}$$

<u>Review of Commutation / Eigenstates:</u>

- We can only measure $\hat{L}^2$ and <u>one</u> $\hat{L}_i$ at a time

(the $\hat{L}_i$'s don't commute - as we'll see)

$$[\hat{L}_i, \hat{L}_j] = \hat{L}_i\hat{L}_j - \hat{L}_j\hat{L}_i$$

$$= \ldots$$

$$= i\,\epsilon_{ijk}\,\hat{L}_k$$

We generally choose $\hat{L}_z$ to be the operator commuting with $\hat{L}^2$.

Since $\hat{L}_x, \hat{L}_y$ won't commute, we can define

$$\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y$$

as "raising/lowering" operators for angular momentum, in such a way that

$$\hat{L}^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) |\ell, m\rangle$$

$$\hat{L}_z |\ell, m\rangle = \hbar m |\ell, m\rangle$$

$\ell = 0, 1, 2, \ldots$

$(\ell \in \mathbb{Z}^{\geq 0})$

$m = -\ell, \ldots, \ell$

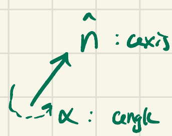
$$\hat{L}_+ |\ell, m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m+1)}\, |\ell, m+1\rangle$$

$$\hat{L}_- |\ell, m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m-1)}\, |\ell, m-1\rangle$$

THUS, for a given ℓ: there are $2\ell+1$ states

$$2\ell+1 \left\{ \begin{array}{l} |\ell,\ell\rangle \; - \quad \overset{0}{\uparrow} \hat{L}_+ \\ |\ell,\ell-1\rangle \; - \\ \quad \vdots \\ |\ell, 0\rangle \; - \qquad \downarrow \hat{L}_- \quad \uparrow \hat{L}_+ \\ \quad \vdots \\ |\ell, -\ell+1\rangle \; - \\ |\ell, -\ell\rangle \; - \quad \underset{0}{\downarrow} \hat{L}_- \end{array} \right.$$

Let's now consider a rotation in 3D about an arbitrary axis $\hat{n}$ by an angle α.



Let $\hat{R}(\alpha, \hat{n}) \overset{\text{def}}{\equiv} e^{-i\frac{\alpha}{\hbar}\hat{n}\cdot\vec{L}}$

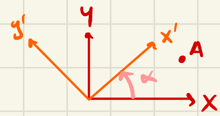
$$\simeq 1 - i\frac{\alpha}{\hbar}\hat{n}\cdot\vec{L} + \ldots$$

note the minus sign...

Spinless particle: Scalar wavefunction.

Let R be a rotation and $\vec{r}_A$ be the point A.

$$\vec{r}_A{}' = R\,\vec{r}_A$$



ψ scalar $\Rightarrow$ $\psi'(\vec{r}\,') = \psi(\vec{r})$

$$\text{so}\quad \psi'(\vec{r}) = \psi(R^{-1}\vec{r})$$
$$= e^{-i\frac{\alpha}{\hbar}\hat{L}_z}\,\psi(\vec{r})$$

Spin-$\frac{1}{2}$ (Fermion) Particle: "Spinor" wavefunction (2-component in NRQM,
4-component 'bispinor' in RQM)

Spin-1 (vector) Particle: vector wavefunction

$\downarrow$ and so on...

(Back to rotations...)

$\rightarrow$ For $l=1$: there are three rotation states

$$\left.\begin{matrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \end{matrix}\right\}$$

$\hat{R}(\alpha,\vec{n})$ mixes these states among each other in a manner that can be determined from the above information.

IN GENERAL: for any $\ell \in \mathbb{Z}^{\geq 0}$, there are $2\ell+1$ states which 'mix' with each other under rotations (rotations do not change ℓ, so different ℓ's do not mix... we say $\{|\ell, m\rangle : m = -\ell, \dots, \ell\}$ transform "irreducibly" under rotations, in that there is no subset of $2\ell+1$ states that is closed under rotations).

For any $\ell \in \mathbb{Z}^{\geq 0}$ there $\exists$ a rotation R such that $|\ell, m\rangle \xrightarrow{R} |\ell, m'\rangle$ for any $m, m' \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$.

* These $2\ell+1$ states $(0, 3, 5, 7, \dots)$ are said to be the "irreducible unitary representations of $SO(3)$": any rotation of $\mathbb{R}^3$ is "represented" by a $(2\ell+1) \times (2\ell+1)$ - dimensional unitary matrix; the product of two rotations is represented by the product of matrices; the inverse rotation is the inverse matrix, ...

But in QM, things change:

$\vec{r}, \vec{p} \longrightarrow$ operators $\hat{\vec{r}}, \hat{\vec{p}}$ acting in Hilbert space $\mathcal{H}$

$\longrightarrow$ the space of states of a physical system,

which is complex $(\mathbb{C}, \mathbb{C}^2, \dots)$

"Spin" or "Intrinsic angular momentum" arises because

of a fundamental relation between "rotations" of $\mathbb{R}^3$ and

"rotations" of a 2-dimensional complex space $\mathbb{C}^2$.

$\mathbb{C}^2$ Algebra

Let $z_1, z_2 \in \mathbb{C}$. Define two vectors $|z\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $|\tilde{z}\rangle = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}$.

Then $\langle \tilde{z} | z \rangle = \bar{\tilde{z}}_1 z_1 + \bar{\tilde{z}}_2 z_2 = (\bar{\tilde{z}}_1\ \bar{\tilde{z}}_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ Inner Product

and $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \mathcal{U}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$; $\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \mapsto \mathcal{U}\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}$

if $\mathcal{U}^\dagger \mathcal{U} = \mathbb{1}_{2\times 2}$ inner product is preserved.

- $\mathcal{U}$ belongs in the group of 2×2 unitary matrices with $\det(\mathcal{U}) = 1$

called $SU(2)$

What does this have to do with anything regarding rotations in $\mathbb{R}^3$?

Consider the space of <u>2×2 $\mathbb{C}$ Hermitian traceless matrices.</u>

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \text{general } \mathbb{C} \text{ matrix}, \quad a,b,c,d \in \mathbb{C}.$$

* $\mathrm{tr}(A) = 0 \implies a = -d.$

$$\implies A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

* $A^\dagger = A \implies \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & -\bar{a} \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$

$$\implies a = \bar{a} \; ; \; b = \bar{c}$$

Can be resolved via $A = \begin{pmatrix} a & f-ig \\ f+ig & -a \end{pmatrix}$ $[a, f, g \in \mathbb{R}]$

Expand:

$$A = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + f\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + g\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

which is true $\forall A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}^2 ; \mathrm{tr}=0)$.

with basis vectors $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\}$.

- Yields Pauli matrices

$$\left.\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}\right\} \text{Basis in 3D space of } 2\times 2 \text{ Hermitian traceless matrices.}$$

and denote $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ collectively.

- Then, as before, $\vec{\sigma}\cdot\vec{a} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$

$$= \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_3 \end{pmatrix}.$$

In particular, let $\vec{a} \in \mathbb{R}^3$. We can take $\vec{a} = \vec{x} = (x, y, z)$ or $(x_1, x_2, x_3), \dots$

and consider the map

$$\vec{x} \in \mathbb{R}^3 \longleftrightarrow \vec{x}\cdot\vec{\sigma} \in \left\{ \begin{matrix} \text{2x2 Hermitian,} \\ \text{traceless, } \mathbb{C} \\ \text{matrices} \end{matrix} \right\}$$

$$= \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

Properties: (1) $\mathrm{tr}(\vec{x}\cdot\vec{\sigma}) = 0$ (obviously)

(2) $\det(\vec{x}\cdot\vec{\sigma}) = -|\vec{x}|^2$

(3) $\sigma_i^2 = \mathbb{1}$ for $i = 1, 2, 3$.

(sometimes people consider the quaternion $i\,\vec{x}\cdot\vec{\sigma} \rightarrow \det(i\,\vec{x}\cdot\vec{\sigma}) = |\vec{x}|^2$)

Consider now the following transformation with an arbitrary $\mathcal{U} \in SU(2)$:

$$\vec{x}\cdot\vec{\sigma} \longrightarrow \mathcal{U}\,\vec{x}\cdot\vec{\sigma}\,\mathcal{U}^\dagger.$$

This transformation preserves both trace and determinant of $\vec{x}\cdot\vec{\sigma}$:

- $\mathrm{tr}(\mathcal{U}\,\vec{x}\cdot\vec{\sigma}\,\mathcal{U}^\dagger) = \mathrm{tr}(\overset{\mathbb{1}}{\cancel{\mathcal{U}\mathcal{U}^\dagger}}\,\vec{x}\cdot\vec{\sigma}) = 0$

- $\det(\mathcal{U}\,\vec{x}\cdot\vec{\sigma}\,\mathcal{U}^\dagger) = \overset{1}{\cancel{\det(\mathcal{U})}}\,\overset{1}{\cancel{\det(\mathcal{U}^\dagger)}}\,\det(\vec{x}\cdot\vec{\sigma})$

which implies

$$\vec{x}\cdot\vec{\sigma} \longmapsto U\,\vec{x}\cdot\vec{\sigma}\,U^{\dagger} \equiv \vec{x}'\cdot\vec{\sigma}$$

$$|\vec{x}'|^2 = |\vec{x}|^2$$

and furthermore, the dependence of $\vec{x}'$ on $\vec{x}$ is linear, so it must be that $\vec{x}\cdot\vec{\sigma} \longrightarrow \vec{x}'\cdot\vec{\sigma} = U\,\vec{x}\cdot\vec{\sigma}\,U^{\dagger}$ is a rotation of $\mathbb{R}^3$ when expressed as a transformation on $\vec{x}$. Since any rotation preserves length.

How to see:

$$\sum_{a=1}^{3} x_a'\sigma_a = \sum_{b=1}^{3} x_b \underbrace{U\sigma_b U^{\dagger}}_{\text{matrix}} \qquad (U \in SU(2))$$

const. (x_b)

omit sums from now on.

Since $U\sigma_b U^{\dagger}$ has zero trace & is Hermitian, it must be a linear combination of σ's (change of basis) so we say that $U\sigma_b U^{\dagger} = R_{bc}\sigma_c$ where R_{bc} is real.

sum c implied

R_{bc}: "change of basis rotation"

So we have

$$x_a'\sigma_a = x_b U\sigma_b U^{\dagger}$$

$$= x_b R_{bc}\sigma_c$$

$$\Longrightarrow \boxed{x_a' = R_{ab}\sigma_b}$$

index change.

Furthermore $u \sigma_b u^{\dagger} = R_{bc} \sigma_c$.

DIGRESSION:

Pauli matrices obey the algebra

$$\sigma_a \sigma_b = \delta_{ab} \mathbb{1} + i \epsilon_{abc} \sigma_c \quad ; \quad a,b = 1,2,3.$$

which implies $\mathrm{tr}(\sigma_a \sigma_b) = \delta_{ab} \, \mathrm{tr}(\mathbb{1})$; $[\mathrm{tr}(\sigma_c) = 0]$

END DIGRESSION.

We can express R_{bc} as follows:

$$u \sigma_b u^{\dagger} = R_{bc} \sigma_c \implies u \sigma_b u^{\dagger} \sigma_d = R_{bc} \sigma_c \sigma_d$$

$$\implies \mathrm{tr}(u \sigma_b u^{\dagger} \sigma_d) = \mathrm{tr}(R_{bc} \sigma_c \sigma_d)$$

$$= R_{bc} \, \mathrm{tr}(\sigma_c \sigma_d)$$

$$= R_{bc} \cdot 2 \delta_{cd}$$

$$= 2 R_{bd}.$$

This immediately implies $R_{bd} = \frac{1}{2} \mathrm{tr}(u \sigma_b u^{\dagger} \sigma_d)$

SO(3) matrix

SU(2) matrices

<u>Claims</u> : ① $\forall u \in SU(2)$, $\exists R \in SO(3)$

② u and $-u \in SU(2)$ gives rise to same $R \in SO(3)$.

$SU(2)$ is "universal cover" of $SO(3)$.

$\therefore$ All rotations in 3D $\mathbb{R}^3$ can also be described by using "rotations" by u in $\mathbb{C}^2$.

* The description is not unique in the sense that $\forall R \to u, -u$ give rise to same R.

Since QM operates in $\mathbb{C}$- Hilbert space, we can ask:

If we have wavefunctions like $|l,m\rangle$ w/ $l \in \mathbb{Z}^{\geq 0}$ which transform as (say, $l=1$) 3x3 matrices under rotation, the structure of 3D space allows us to have wave functions that transform like

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \xleftrightarrow[R]{\text{rotation}} \pm u \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

sign is irrelevant ; isn't an observable of ψ.

As usual in QM, if something is allowed by symmetries, it exists.

Since wavefunctions like $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto u \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ under 3D rotations are permitted, they were found to exist.